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Estimating Eigenfunctions of the Infinitesimal Generator of a Stochastic Differential Equation as Solutions of Integral Equations

Ruhi Tuncer

GALATASARAY UNIVERSITY
ECONOMIC RESEARCH CENTER

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Ruhi Tuncer
Department of Economics
Galatasaray University
Istanbul Turkey

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Abstract

We propose to estimate the eigenfunctions of the infinitesimal generator of a stochastic differential equation as the solutions of an integral equation.

1 Introduction

Eigenfunctions of the infinitesimal generator of a stochastic differential equation have been used in identifying the their diffusion and drift functions. Let

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t$$

be a time homogeneous stochastic differential equation where $\mu(X_t)$ is the drift and $\sigma(X_t)$ is the diffusion.

Let $\phi(X_t)$ be a twice continuously differentiable function with

$E[\phi(X_t)] = 0$ and $E[\phi^2(X_t)] < \infty$ where the expected values are taken with respect to the stationary distribution $q(X_t)$ of X_t Let

$$A\phi = lim_{t\to 0}\frac{E[\phi(X_t)]X_0] - \phi(X_0)}{t}$$

be the infinitesimal generator of the process X_t.

The infinitesimal generator satisfies Kolmogorov's backward equation:

$$\frac{\partial \phi(t,x)}{\partial t} = A\phi(t,x)$$

i.e.

$$\frac{\partial \phi(t,x)}{\partial t} = \mu(x)\frac{\partial \phi(t,x)}{\partial x} + \frac{1}{2}\sigma^2(x)\frac{\partial^2 \phi(t,x)}{\partial x^2}.$$

Separation of variables gives

$$\mu(x)\frac{\partial \phi}{\partial x} + \frac{1}{2}\sigma^2(x)\frac{\partial^2 \phi}{\partial x^2} = \lambda\phi(x)$$

and

$$\phi(t,x) = exp(\lambda t)\phi(x)$$

So that

$$E[\phi(X_t)|X_0 = x] = exp(\lambda t)\phi(x).$$

2 Integral Equations and Eigenfunction Estimation

A homogeneous Fredholm integral equation of the second kind is an equation that has the following form:

$$\alpha \int_R \phi(y)K(x,y)dy = \phi(x) \tag{1}$$

where $K(x,y)$ is called the kernel. Complicated analytic and numerical techniques exist for solving these equations. Fortunately a combination of econometric and numerical techniques exists, enabling us to use them for estimating the eigenfunctions of the infinitesimal generator of a stochastic differential equation.

Suppose that we are given the estimate of the transition density $p(x,y)$ of the underlying variable X_t.

$$E[\phi(X_t)|X_{t-1}] = exp(\lambda)\phi(X_{t-1})$$

Let $\rho = exp(\lambda)$, $x = X_{t-1}$ and $y = X_t$ so that we have

$E[\phi(y)|x] = \rho\phi(x)$ i.e.

$$\int_R \phi(y)p(x,y)dy = \rho\phi(x) \quad (2)$$

This is a homogeneous Fredholm integral equation of the second kind. We will follow Fredholm's method and replace the integral with a Riemann sum.

$$\sum_{j=1}^{n} \phi(y_j)p(x_i, y_j) = \rho\phi(x_i) \quad (3)$$

for $i = 1, \dots, n$.

Let $P = [p(x_i, y_j)]$ be the matrix of transition probabilities. $x = [x_i]$ is the vector of points and $\phi(x)$ is the vector of functions evaluated at these points.

We have

$$P\phi(x) = \rho\phi(x) \quad (4)$$

i.e.

$$(\rho I - P)\phi(x) = 0 \quad (5)$$

The solution of this eigenvalue problem will give us the values of $\phi(x)$.

In a Fredholm equation of the second kind, α thus ρ is given and this is problematic if ρ is not an eigenvalue of the matrix. However In our case ρ is unknown and will be calculated.

Sturm-Liouville theory imposes certain constraints on the solutions. We know that the infinitesimal generator of a diffusion on a compact interval with reflective barriers $[l, r]$, is negative semidefinite operator.

$$A\phi = -\lambda\phi$$

with $\lambda > 0$ and $\phi'(l) = \phi'(r) = 0$.

Since $\rho = exp(\lambda)$, constraints are imposed on the eigenvalues of the transition probability matrix P. The eigenvalues should in the interval $0 < \rho < 1$. This is the reason why we can not use recursive techniques to solve the integral equation.

The solution of this eigenvalue eigenvector problem gives us a discrete approximation to the eigenfunction. We therefore have to find a continuous interpolating function for smoothing. The logistic function seems to be a good candidate for the job.

We will use

$$\phi(x) = \frac{1}{1+e^{-ax}} \tag{6}$$

as the smoothing function.

3 Drift and Diffusion as Functions of the Transition Probabilities

If we are lucky enough to find two eigenvalues satisfying the conditions, we can use the method proposed by Demoura (1998) to obtain estimates of the drift $\mu(x)$ and the diffusion $\sigma(x)$.

Let ρ_1 and ρ_2 be the two appropriate eigenvalues

Let ϕ_1 and ϕ_2 be two eigenfunctions associated with eigenvalues ρ_1 and ρ_2.

Remembering that $\rho_1 = exp(\lambda_1)$ and $\rho_2 = exp(\lambda_2)$ we have

$$\mu\phi_1' + 1/2\sigma^2\phi_1'' = \lambda_1\phi_1$$

and

$$\mu\phi_2' + 1/2\sigma^2\phi_2'' = \lambda_2\phi_2.$$

Solving these two equations simultaneously will give us the estimates of the drift and the diffusion functions.

References

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