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The Qualitative Study of the Transitional Dynamics of the Romer Model

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ABSTRACT. This paper applies the method of Barro and Sala-i Martin to the stability analysis of the Romer model and verifies sufficient conditions for its application.

KEY WORDS: Endogenous growth, Stability, Transitional dynamics.

JEL CLASSIFICATION: 041.

INTRODUCTION: EQUILIBRIUM GROWTH RATES.

Arnold (2000) points out that the Romer model becomes equivalent to the Uzawa-Lucas model when $\beta = 0$. Then, the method of Barro and Sala-i Martin (1995, p.204) can be employed to study the global stability of the Romer model. This paper will effectuate the stability analysis of the Romer model by the method of Barro and Sala-i Martin and investigate whether there are other conditions, than the assumption $\beta = 0$, that make possible the application of the aforementioned method and finally verify the transversality conditions wrt the two state variables.

The Romer model consists of the following maximization problem:
$\max \int_0^\infty \frac{C^{1-\theta}}{1-\theta} e^{-\rho t} dt$

$$\begin{aligned} \dot{A} &= \sigma H_A A 1 \quad (1) \\ \dot{K} &= Y - C - \delta K 2 \quad (2) \end{aligned}$$

$$\begin{aligned} A_0 &= A(0) > 0, \quad K_0 = K(0) > 0 \\ t \to \infty \lim A &\geq 0, \ \ t \to \infty \lim K \geq 0 \end{aligned}$$

Where $Y = F(A, K, L, H_A) = \gamma^{\alpha+\beta-1} A^{\alpha+\beta} (H - H_A)^{\alpha} L^{\beta} K^{1-\alpha-\beta}$; which, assuming L constant and defining $\Upsilon \equiv \gamma^{\alpha+\beta-1} L^{\beta}$, can be expressed as

$$Y = F(A, K, H_A) = \Upsilon A^{\alpha+\beta} (H - H_A)^{\alpha} K^{1-\alpha-\beta}.3 \quad (3)$$

The current-value hamiltonian is H=C$^{1-\theta}\frac{}{1-\theta+\mu(\sigma H_A A)+\lambda(Y-C-\delta K),}$and the first order conditions are

$$\begin{aligned}
\frac{\partial \mathcal{H}}{\partial C} &= 0 \Longrightarrow \; C^{-\theta} = \lambda 4 && (4)\\
\frac{\partial \mathcal{H}}{\partial H_A} &= 0 \Longrightarrow \; \frac{\lambda}{\mu} = \frac{\sigma A}{\alpha Y}\left(H - H_A\right) 5 && (5)\\
\frac{\partial \mathcal{H}}{\partial A} &= -\dot{\mu} + \mu\rho \Longrightarrow \dot{\mu} = -\mu\sigma H_A - \lambda F_A + \mu\rho \Longrightarrow \dot{\mu} = -\mu\sigma H_A - \lambda\left(\alpha+\beta\right) Y A^{-1} + \mu\rho 6 && (6)\\
\frac{\partial \mathcal{H}}{\partial K} &= -\dot{\lambda} + \lambda\rho \Longrightarrow \dot{\lambda} = -\lambda\left[F_K - \delta\right] + \lambda\rho \Longrightarrow \dot{\lambda} = -\lambda\left[\left(1-\alpha-\beta\right) Y K^{-1} - \delta\right] + \lambda\rho 7 && (7)
\end{aligned}$$

Assume K grows with a constant rate, then $I = S$ iff Y grows with the same constant rate. Then C also grows with the same constant rate. And quoting Romer, "The intuition from the Solow model suggests that ... an equilibrium will exist if A grows at a constant exponential rate ... A will grow at a constant rate if H_A ...stays constant" (1990, p.S90). Thus, L being constant, H_A remaining constant by intuition, A grows with the same constant rate than K —since (3) is homogeneous of degree one wrt K and A— :

$$\frac{\dot{Y}}{Y} = \frac{\dot{K}}{K} = \frac{\dot{C}}{C} = \frac{\dot{A}}{A} = \sigma H_A .8 \qquad (8)$$

Then, from (4) and (8) we obtain

$$\frac{\dot{\lambda}}{\lambda} = -\theta\sigma H_A .9 \qquad (9)$$

and from (5) and (6) we obtain

$$\begin{aligned}
\frac{\dot{\mu}}{\mu} &= -\sigma H_A - \frac{\sigma A}{\alpha Y}\left(H - H_A\right)\left(\alpha+\beta\right) Y A^{-1} + \rho && (10)\\
&= -\sigma\left[-\frac{\beta}{\alpha} H_A + \frac{\alpha+\beta}{\alpha} H\right] + \rho .10 && (11)
\end{aligned}$$

Since by intuition H_A is constant, and from (8) A/Y is constant then λ/μ is constant in (5), then we can equalize (9) and (10) to obtain the equilibrium value of H_A

$$H_A^* = \frac{\sigma\left(\alpha+\beta\right) H - \alpha\rho}{\sigma\left(\alpha\theta+\beta\right)}, 11 \qquad (12)$$

so the intuition proves true. And the equilibrium growth rates are $*\dot{Y}/Y = *\dot{C}/C =$

$*\dot{A}/A = *\dot{K}/K = \sigma H_A^*$.

EXISTENCE, UNICITY, STABILITY.

To prove the existence of the equilibrium we have to solve the system (1)-(7), for the policy variables C and H_A —which is already found in (11)— , for the state variables K and A, for the costate variables μ and λ, and for Y, in all for seven variables. To reduce the system, define $\chi \equiv C/K$ and $\omega \equiv K/A$. Using (2) and (3) we have

$$\frac{\dot{K}}{K} = \Upsilon \left(H - H_A\right)^{\alpha} \omega^{-(\alpha+\beta)} - \chi - \delta .12 \tag{13}$$

And using (1) and (12) we get

$$\frac{\dot{\omega}}{\omega} = \frac{\dot{K}}{K} - \frac{\dot{A}}{A} \tag{14}$$

$$= \Upsilon \left(H - H_A\right)^{\alpha} \omega^{-(\alpha+\beta)} - \chi - \delta - \sigma H_A .13 \tag{15}$$

From (7), using (3) we have

$$\frac{\dot{\lambda}}{\lambda} = -\left(1 - \alpha - \beta\right) \Upsilon \left(H - H_A\right)^{\alpha} \omega^{-(\alpha+\beta)} + \delta + \rho .14 \tag{16}$$

(4) and (14) give

$$\frac{\dot{C}}{C} = \frac{1}{\theta} \left[\left(1 - \alpha - \beta\right) \Upsilon \left(H - H_A\right)^{\alpha} \omega^{-(\alpha+\beta)} - \delta - \rho\right], 15 \tag{17}$$

and (12) and (15) together is

$$\frac{\dot{\chi}}{\chi} = \frac{\dot{C}}{C} - \frac{\dot{K}}{K} \tag{18}$$

$$= \frac{1 - \alpha - \beta - \theta}{\theta} \Upsilon \left(H - H_A\right)^{\alpha} \omega^{-(\alpha+\beta)} + \chi - \frac{\left(1 - \theta\right) \delta + \rho}{\theta} .16 \tag{19}$$

Logarithmic differentiation of (5) gives

$$\frac{\cdot(\mu/\lambda)}{\mu/\lambda} = \frac{\alpha}{\sigma} \Upsilon [- \left(\alpha - 1\right) \left(H - H_A\right)^{\alpha-2} \dot{H}_A \omega^{1-\alpha-\beta} \tag{20}$$

$$+ \left(H - H_A\right)^{\alpha-1} \left(1 - \alpha - \beta\right) \omega^{-(\alpha+\beta)} \dot{\omega}] \tag{21}$$

$$\div \frac{\alpha}{\sigma} \Upsilon \left(H - H_A\right)^{\alpha-1} \omega^{1-\alpha-\beta} \tag{22}$$

$$= \quad -(\alpha - 1)\frac{\dot{H}_A}{(H - H_A)} + (1 - \alpha - \beta)\frac{\dot{\omega}}{\omega}.17 \tag{23}$$

On the other hand from (6), using (3) we have

$$\frac{\dot{\mu}}{\mu} = -\sigma H_A - \frac{\lambda}{\mu}(\alpha + \beta)\Upsilon (H - H_A)^{\alpha}\omega^{1-\alpha-\beta} + \rho.18 \tag{24}$$

Taking into account that $\lambda/\mu = \sigma/\alpha\Upsilon (H - H_A)^{\alpha-1}\omega^{1-\alpha-\beta}$ and combining (14), (17) and (18) we get

$$-(\alpha - 1)\frac{\dot{H}_A}{(H - H_A)} + (1 - \alpha - \beta)\frac{\dot{\omega}}{\omega} \tag{25}$$

$$= \quad -\sigma H_A - \frac{\sigma}{\alpha}(\alpha + \beta)(H - H_A) \tag{26}$$

$$+ (1 - \alpha - \beta)\Upsilon (H - H_A)^{\alpha}\omega^{-(\alpha+\beta)} - \delta.19 \tag{27}$$

Let replace (13) into (19) to obtain

$$\frac{\dot{H}_A}{(H - H_A)} \quad = \quad \frac{1 - \alpha - \beta}{1 - \alpha}\chi - \frac{\alpha + \beta}{1 - \alpha}\delta \tag{28}$$

$$-\frac{\alpha + \beta}{1 - \alpha}\sigma H_A - \frac{\sigma}{\alpha}\frac{\alpha + \beta}{1 - \alpha}(H - H_A).20 \tag{29}$$

Thus the system is reduced to three equations (20), (16), (13) for three variables ω, χ and H_A. Defining

$$z \equiv \frac{Y}{K} = \Upsilon (H - H_A)^{\alpha}\omega^{-(\alpha+\beta)}21 \tag{30}$$

in (16) and (13) the system becomes

$$\frac{\dot{H}_A}{(H - H_A)} \quad = \quad \frac{1 - \alpha - \beta}{1 - \alpha}\chi - \frac{\alpha + \beta}{1 - \alpha}\delta \tag{31}$$

$$-\frac{\alpha + \beta}{1 - \alpha}\sigma H_A - \frac{\sigma}{\alpha}\frac{\alpha + \beta}{1 - \alpha}(H - H_A)\,20 \tag{32}$$

$$\frac{\dot{\chi}}{\chi} \quad = \quad \frac{1 - \alpha - \beta - \theta}{\theta}z + \chi - \frac{(1 - \theta)\delta + \rho}{\theta}22 \tag{33}$$

$$\frac{\dot{\omega}}{\omega} \quad = \quad z - \chi - \delta - \sigma H_A 23 \tag{34}$$

whose solution, after setting time derivatives equal to zero, proves the existence and unicity of the equilibrium [appendix]:

$$H_A^* = \frac{(\alpha+\beta)\,\sigma H - \alpha\rho}{[\beta+\alpha\theta]\,\sigma} 24.1 \tag{35}$$

$$\chi^* = \frac{\alpha+\beta}{1-\alpha-\beta} \cdot \frac{(\beta+\alpha\theta)\,\delta + (\theta+\alpha+\beta-1)\,\sigma H + (1-\alpha)\,\rho}{(\beta+\alpha\theta)} 24.2 \tag{36}$$

$$z^* = \frac{(\alpha+\beta)\,\sigma\theta H + (\beta+\alpha\theta)\,\delta + \beta\rho}{(1-\alpha-\beta)\,(\beta+\alpha\theta)} 24.3 \tag{37}$$

The system (20), (22), (23) can be expressed as differences of the variables from their equilibrium values [appendix], which is a suitable form for the study of stability:

$$\frac{\dot{H}_A}{H-H_A} = \frac{(1-\alpha-\beta)}{1-\alpha}(\chi-\chi^*) + \frac{\alpha+\beta}{\alpha}\sigma\,(H_A - H_A^*)\,25 \tag{38}$$

$$\frac{\dot{\chi}}{\chi} = \frac{1-\alpha-\beta-\theta}{\theta}(z-z^*) + (\chi-\chi^*)\,26 \tag{39}$$

$$\frac{\dot{\omega}}{\omega} = (z-z^*) - (\chi-\chi^*) - \sigma\,(H_A - H_A^*)\,27 \tag{40}$$

Replacing (25) and (27) into (21) provides a forth equation:

$$\frac{\dot{z}}{z} = -(\alpha+\beta)(z-z^*) + \frac{\beta}{1-\alpha}(\chi-\chi^*)\,.28 \tag{41}$$

The method of Barro and Sala-i Martin consists in finding among (25)-(28) an equation which expresses the growth rate of one of the variables as function of itself alone. The observation reveals two appropriate equations among them: the eq. (26) for the case $1-\alpha-\beta-\theta=0$; and the eq. (28) for the case $\beta=0$. In the case $1-\alpha-\beta-\theta=0$, according to the eq. (26), which becomes $\dot{\chi}/\chi = \chi - \chi^*$, χ does not converge to its equilibrium value χ^*. Thus the remaining case is the case $\beta=0$, which simplifies the eqs. (25)-(28) into

$$\frac{\dot{\omega}}{\omega} = (z-z^*) - (\chi-\chi^*) - \sigma\,(H_A - H_A^*)\,29 \tag{42}$$

$$\frac{\dot{\chi}}{\chi} = \frac{1-\alpha-\theta}{\theta}(z-z^*) + (\chi-\chi^*)\,30 \tag{43}$$

$$\frac{\dot{H}_A}{H-H_A} = (\chi-\chi^*) + \sigma\,(H_A - H_A^*)\,31 \tag{44}$$

$$\frac{\dot{z}}{z} = -\alpha\,(z-z^*)\,.32 \tag{45}$$

The solution of (29)-(32) gives the equilibrium values,

$$H_A^* = \frac{\sigma H - \rho}{\sigma\theta} 33.1 \tag{46}$$

$$\chi^* = \frac{\alpha\theta\delta + (\theta + \alpha - 1)\,\sigma H + (1-\alpha)\,\rho}{(1-\alpha)\,\theta} 33.2 \tag{47}$$

$$z^* = \frac{\theta H + \delta}{1-\alpha}.33.3 \tag{48}$$

And the equilibrium growth rate is

$$*\dot{K}/K = \frac{\sigma H - \rho}{\theta}.34 \tag{49}$$

Now we can proceed to the stability analysis which is global —since not restricted to the neighborhood of the equilibrium point— : the integral and the solution of (32) are

$$\frac{z - z^*}{z} = \left(\frac{z_0 - z^*}{z_0}\right) e^{-\alpha z^* t} \tag{50}$$

$$z = \frac{z^* z_0}{(z^* - z_0)\, e^{-\alpha z^* t} + z_0} 35 \tag{51}$$

where $z_0 = z\,(0) = Y\,(0)\,/K\,(0) > 0$. The time derivative of (35) is

$$\dot{z} = \frac{(z^* - z_0)\,\alpha z^* e^{-\alpha z^* t} z^* z_0}{\left[(z^* - z_0)\, e^{-\alpha z^* t} + z_0\right]^2}.36 \tag{52}$$

(36) and (37) imply that $t \longrightarrow \infty \Longrightarrow z \longrightarrow z^*$ and $\dot{z} \longrightarrow 0$; and that this approach is monotonic: $z_0 z^* \Longrightarrow \dot{z} 0$ and $z - z^* 0$ for all t.

Now we can study the behavior of χ from (30); six cases can arise:

1) $(1-\alpha-\theta)\,/\theta > 0$ and $z - z^* > 0$. If $\chi - \chi^* \geq 0$ then $\dot{\chi} > 0$, χ moves away from χ^*, stability requires $\chi - \chi^* < 0$. If $\chi - \chi^* < 0$ and $\dot{\chi} < 0$, then χ moves away from χ^*, stability requires $\dot{\chi} > 0$.Hence for stability $\chi - \chi^* < 0$ and $\dot{\chi} > 0$.

2) $(1-\alpha-\theta)\,/\theta > 0$ and $z - z^* < 0$. If $\chi - \chi^* \leq 0$ then $\dot{\chi} < 0$, χ moves away from χ^*, stability requires $\chi - \chi^* > 0$. If $\chi - \chi^* > 0$ and $\dot{\chi} > 0$, then χ moves away from χ^*, stability requires $\dot{\chi} < 0$.Hence for stability $\chi - \chi^* > 0$ and $\dot{\chi} < 0$.

3) $(1-\alpha-\theta)\,/\theta = 0$ and $z - z^* 0$. If $\chi - \chi^* > 0$ then $\dot{\chi} > 0$, so χ moves away from χ^* that is why the stability requires $\chi - \chi^* < 0$.If $\chi - \chi^* < 0$ then $\dot{\chi} < 0$, χ moves away from χ^*. Hence for stability $\chi = \chi^*$ and $\dot{\chi} = 0$.

4) $(1-\alpha-\theta)\,/\theta < 0$ and $z - z^* > 0$. If $\chi - \chi^* \leq 0$ then $\dot{\chi} < 0$, so χ moves away from

χ^* that is why the stability requires $\chi - \chi^* > 0$. If $\chi - \chi^* > 0$ and $\dot{\chi} > 0$, then χ moves away from χ^* that is why stability needs $\dot{\chi} < 0$.Hence for stability $\chi - \chi^* > 0$ and $\dot{\chi} < 0$.

5) $(1-\alpha-\theta)/\theta < 0$ and $z - z^* < 0$. If $\chi - \chi^* \geq 0$ then $\dot{\chi} > 0$, so χ moves away from χ^* that is why the stability requires $\chi - \chi^* < 0$. If $\chi - \chi^* < 0$ and $\dot{\chi} < 0$, then χ moves away from χ^* that is why stability needs $\dot{\chi} > 0$.Hence for stability $\chi - \chi^* < 0$ and $\dot{\chi} > 0$.

6) $(1-\alpha-\theta)/\theta 0$ and $z - z^* = 0$. If $\chi - \chi^* > 0$ then $\dot{\chi} > 0$, so χ moves away from χ^* that is why the stability requires $\chi - \chi^* < 0$.If $\chi - \chi^* < 0$ then $\dot{\chi} < 0$, χ moves away from χ^* . Hence for stability $\chi = \chi^*$ and $\dot{\chi} = 0$.

Now we determine the stability path of H_A. Three possible cases to examine follow from (31):

i) $\chi - \chi^* > 0$. If $H_A - H_A^* \geq 0$ then $\dot{H}_A > 0$, H_A moves away from H_A^* towards H, stability requires $H_A - H_A^* < 0$. If $H_A - H_A^* < 0$ and $\dot{H}_A < 0$ then H_A moves away from H_A^* towards zero, stability requires $\dot{H}_A > 0$. For stability $H_A - H_A^* < 0$ and $\dot{H}_A > 0$.

ii) $\chi - \chi^* < 0$. If $H_A - H_A^* \leq 0$ then $\dot{H}_A < 0$, H_A moves away from H_A^* towards zero, stability requires $H_A - H_A^* > 0$. If $H_A - H_A^* > 0$ and $\dot{H}_A > 0$ then H_A moves away from H_A^* towards H_A^*, stability requires $\dot{H}_A < 0$. For stability $H_A - H_A^* > 0$ and $\dot{H}_A < 0$.

iii) $\chi - \chi^* = 0$. If $H_A - H_A^* > 0$ then $\dot{H}_A > 0$, H_A moves away from H_A^* towards H, stability requires $H_A - H_A^* < 0$. If $H_A - H_A^* < 0$ then $\dot{H}_A < 0$, H_A moves away from H_A^* towards zero, stability requires $H_A = H_A^*$ and then $\dot{H}_A = 0$.

Combining (29) and (30), and combining (29) and (31) we obtain respectively

$$\frac{\dot{\omega}}{\omega} = \frac{1-\alpha}{\theta}(z - z^*) - \frac{\dot{\chi}}{\chi} - \sigma(H_A - H_A^*)\, 37 \tag{53}$$

and

$$\frac{\dot{\omega}}{\omega} = (z - z^*) - \frac{\dot{H}_A}{H - H_A}, 38 \tag{54}$$

to study the behavior of ω. Six cases arise:

1) $(1-\alpha-\theta)/\theta > 0$ and $z - z^* > 0 \Longrightarrow \chi - \chi^* < 0 \Longrightarrow \dot{H}_A < 0$, and from (38) $\dot{\omega} > 0$.

2) $(1-\alpha-\theta)/\theta > 0$ and $z - z^* < 0 \Longrightarrow \chi - \chi^* > 0 \Longrightarrow \dot{H}_A > 0$, and from (38) $\dot{\omega} < 0$.

3)$(1-\alpha-\theta)/\theta = 0$ and $z - z^* 0 \Longrightarrow \chi = \chi^*$ and $\dot{\chi} = 0 \Longrightarrow H_A = H_A^*$ and then $\dot{H}_A = 0$, and from (37) or (38) $\dot{\omega} 0$.

4)$(1-\alpha-\theta)/\theta < 0$ and $z - z^* > 0 \Longrightarrow \chi > \chi^*$ and $\dot{\chi} < 0 \Longrightarrow H_A - H_A^* < 0$ and $\dot{\chi} < 0$, hence from (37) $\dot{\omega} > 0$.

5)$(1-\alpha-\theta)/\theta < 0$ and $z - z^* < 0 \Longrightarrow \chi < \chi^*$ and $\dot{\chi} > 0 \Longrightarrow H_A - H_A^* > 0$and $\dot{\chi} > 0$, hence from (37) $\dot{\omega} < 0$.

6)$(1-\alpha-\theta)/\theta 0$ and $z-z^*=0 \Longrightarrow \chi=\chi^*$ and $\dot{\chi}=0 \Longrightarrow H_A=H_A^*$ and then $\dot{H}_A=0$, and from (37) or (38) $\dot{\omega}=0$.

Since $\dot{\omega}0$ requires $\omega-\omega^*0$ for stability, we conclude that $z-z^*0$ corresponds to $\omega-\omega^*0$.

We can express the growth rates of the variables C, K, A and Y in terms of the equilibrium growth rate of K and differences of the variables z and χ from their equilibrium values. For the case $\beta=0$, setting $\Upsilon=\gamma^{\alpha-1}$, the production is

$$Y=\Upsilon A^{\alpha}\left(H-H_A\right)^{\alpha} K^{1-\alpha}, 39 \tag{55}$$

and the average product of capital is

$$z=\Upsilon\left(H-H_A\right)^{\alpha} \omega^{-\alpha}.40 \tag{56}$$

(40) and (15) imply that the growth rate of consumption is

$$\frac{\dot{C}}{C}=\frac{1}{\theta}\left[(1-\alpha) z-\delta-\rho\right].41 \tag{57}$$

From (41) and (30) we obtain $\dot{K}_{\overline{K=\frac{1}{\theta}[(1-\alpha)z^*-\delta-\rho]+(z-z^*)-(\chi-\chi^*)}}$ using (33.3)

$$\begin{aligned}\frac{\dot{K}}{K} &= \frac{1}{\theta}\left[(1-\alpha)\frac{\theta H+\delta}{1-\alpha}-\delta-\rho\right]+(z-z^*)-(\chi-\chi^*) \\ &= \frac{1}{\theta}(\theta H-\rho)+(z-z^*)-(\chi-\chi^*).\end{aligned}$$

and replacing (34)

$$\frac{\dot{K}}{K}=*\dot{K}/K+(z-z^*)-(\chi-\chi^*).42 \tag{58}$$

Using (42) and (38)

$$\begin{aligned}\frac{\dot{A}}{A} &= \frac{\dot{K}}{K}-\frac{\dot{\omega}}{\omega} \\ &= *\dot{K}/K+(z-z^*)-(\chi-\chi^*)-\left[(z-z^*)-\frac{\dot{H}_A}{H-H_A}\right] \\ &= *\dot{K}/K-(\chi-\chi^*)+\frac{\dot{H}_A}{H-H_A}\end{aligned}$$

and introducing (31)

$$\frac{\dot{A}}{A}=*\dot{K}/K+\sigma\left(H_A-H_A^*\right).43 \tag{59}$$

In terms of growth rates (39) is $\dot{Y}\overline{{}_{Y=\alpha\frac{\dot{A}}{A}+\alpha\left(-\frac{\dot{H}_A}{H-H_A}\right)+(1-\alpha)\frac{\dot{K}}{K}}}$replacing (31), (42) and (43) $\dot{Y}\overline{{}_{Y=*\dot{K}/K+(1-\alpha)(z-z^*)-(\chi-\chi^*).}}$

Finally we have to check the transversality conditions. The one wrt K is $t\to\infty\lim\left\{-\lambda\cdot e^{-\rho t}\cdot K_{opt}\right\}$ 0. Where, $K_{opt}=K_0\exp\left[\left(\dot{K}/K\right)t\right]$ is the path which begins at $K_0>0$ and moves by the rate (42) to approach assymptotically to the rate $*\dot{K}/K$; and where, from (7), $\lambda=\lambda_0\exp\left[{}_0^t-\left(F_K-\delta-\rho\right)d\tau\right]$ with $\lambda_0>0$ because of (4), and with $F_K=(1-\alpha)z$ approaching assymptotically $F_K^*=(1-\alpha)z^*$; thus, the condition becomes

$$\begin{aligned} t\to\infty\lim\left\{-\lambda_0 e^{{}_0^t-(F_K-\delta-\rho)d\tau}\cdot e^{-\int_0^t\rho d\tau}\cdot K_0 e^{\int_0^t\left(\dot{K}/K\right)d\tau}\right\} &= 0\\ t\to\infty\lim\left\{-\lambda_0 e^{{}_0^t-\left(F_K-\delta-\dot{K}/K\right)dt}K_0\right\} &= 0\\ {}_0^\infty\left(F_K-\delta-\dot{K}/K\right)dt &\to \infty\\ F_K^*-\delta-*\dot{K}/K &> 0. \end{aligned}$$

In other terms, the transversality condition means that at the steady state, the net profit must exceed the growth rate of K. Now we have to check whether the tranversality condition together with the condition $*\dot{K}/K>0$ ensure positive and feasible equilibrium values, namely, whether $H_A^*>0$, $\chi^*>0$, $z^*>0$ and $H_A^*<H$. z^* is obviously positive. H_A^* is positive since $*\dot{K}/K>0$; and it is feasible H-$\text{H}_A^*=\frac{\rho+H(-1+\theta)\sigma}{\sigma\theta}>0$since $F_K^*-\delta-*\dot{K}/K>0$. On the other hand, $F_K^*-\delta-*\dot{K}/K>0$ and $\chi^*-\left(F_K^*-\delta-*\dot{K}/K\right)=\frac{\alpha(\delta+\sigma H)}{1-\alpha}>0$imply together that $\chi^*>0$.

The transversality conditions wrt A is $\lim\left\{-\mu\cdot e^{-\rho t}\cdot A_{opt}\right\}\geq 0$. Where $A_{opt}=A_0\exp\left[\left(\dot{A}/A\right)t\right]$ is the path that begins at $A_0>0$ and moves by the rate (43) to approach to the rate $*\dot{K}/K$; and where, from (6), $\mu=\mu_0\exp\left[{}_0^t\left(-\sigma H_A-(\lambda/\mu)F_A+\rho\right)d\tau\right]$, with $\mu_0>0$ because of (5). Using (10) this condition becomes t$\to\infty\lim\left\{-\mu_0 e^{{}_0^t\left\{-\sigma\left[-\frac{\beta}{\alpha}H_A+\frac{\alpha+\beta}{\alpha}H\right]+\rho\right\}d\tau}\cdot e^{-{}_0^t\rho d\tau}\cdot A_0 e^{{}_0^t\left(\dot{A}/A\right)d\tau}\right\}=$ $0$and using (1),

$$\begin{aligned} t\to\infty\lim\left\{-\mu_0 e^{{}_0^t\sigma\frac{\alpha+\beta}{\alpha}[H_A-H]d\tau}A_0\right\} &= 0\\ {}_0^\infty\sigma\frac{\alpha+\beta}{\alpha}\left[H_A-H\right]d\tau &\to -\infty\\ \sigma\frac{\alpha+\beta}{\alpha}\left[H_A^*-H\right] &< 0 \end{aligned}$$

$$H_A^* < H$$

which is compatible with the result of the previous transversality condition.

CONCLUSION.

For the Romer model, the two cases that allow the stability analysis by the method of Barro and Sala-i Martin are the case $1-\alpha-\beta-\theta=0$ and the case $\beta=0$. While the former does not ensure the convergence , the later provides a stable equilibrium point which is also positive and feasible.

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APPENDIX

In equilibrium the system (20), (22), (23) is

$$\frac{1-\alpha-\beta}{1-\alpha}\chi-\frac{\alpha+\beta}{1-\alpha}\delta-\frac{\alpha+\beta}{1-\alpha}\sigma H_A-\frac{\sigma}{\alpha}\frac{\alpha+\beta}{1-\alpha}\left(H-H_A\right) = 0A1 \tag{60}$$

$$\frac{1-\alpha-\beta-\theta}{\theta}z+\chi-\frac{\left(1-\theta\right)\delta+\rho}{\theta} = 0A2 \tag{61}$$

$$z-\chi-\delta-\sigma H_A = 0A3 \tag{62}$$

Defining $D\equiv\frac{\alpha+\beta}{1-\alpha-\beta}$, $E\equiv\frac{1-\alpha-\beta-\theta}{\theta}$, $1+E\equiv\frac{1-\alpha-\beta}{\theta}$, from (A1) we have

$$\chi=D\delta+D\sigma H_A+\frac{\sigma}{\alpha}D\left(H-H_A\right)A4 \tag{63}$$

and from (A2) we have

$$Ez+\chi-\frac{\left(1-\theta\right)\delta+\rho}{\theta}=0.A5 \tag{64}$$

Combining (A3) and (A5)

$$0=-\left(1+E\right)\chi-E\delta-E\sigma H_A+\frac{\left(1-\theta\right)\delta+\rho}{\theta}.A6 \tag{65}$$

Replacing (A4) into (A6) we have 0=-$\left(1+E\right)D\left(\delta+\sigma H_A+\frac{\sigma}{\alpha}\left(H-H_A\right)\right)-E\delta-E\sigma H_A+$

$\frac{(1-\theta)\delta+\rho}{\theta}$whose solution, using the definitions of D and E, is

$$H_A^* = \frac{(\alpha+\beta)\,\sigma H - \alpha\rho}{(\beta+\alpha\theta)\,\sigma} 24.1 \tag{66}$$

Using the definition of D and (24.1) we obtain from (A4) the 1st expression of χ^*

$$\chi^* = \frac{\alpha+\beta}{1-\alpha-\beta}\left[\delta + \frac{\sigma}{\alpha}H + \frac{\alpha-1}{\alpha}\sigma H_A^*\right] A7 \tag{67}$$

and thence equilibrium value of χ

$$\chi^* = \frac{\alpha+\beta}{1-\alpha-\beta}\cdot\frac{(\beta+\alpha\theta)\,\delta + (\theta+\alpha+\beta-1)\,\sigma H + (1-\alpha)\,\rho}{(\beta+\alpha\theta)} 24.2 \tag{68}$$

Replacing (24.1) and (24.2) in (A3) we find the equilibrium value of z

$$z^* = \frac{(\alpha+\beta)\,\sigma\theta H + (\beta+\alpha\theta)\,\delta + \beta\rho}{(1-\alpha-\beta)\,(\beta+\alpha\theta)} 24.3 \tag{69}$$

From (A2) and (A3) we get respectively the 2nd and the 3rd expressions of χ^*

$$\chi^* = \frac{(1-\theta)\,\delta+\rho}{\theta} - \frac{1-\alpha-\beta-\theta}{\theta} z^* A8 \tag{70}$$

$$\chi^* = z^* - \delta - \sigma H_A^* A9 \tag{71}$$

Now we can express the eqs. (20), (22) and (23) in terms of the differences $(H_A - H_A^*)$, $(\chi-\chi^*)$ and $(z-z^*)$. First, consider (20):

$$\frac{\dot{H}_A}{H-H_A} = \frac{(1-\alpha-\beta)}{1-\alpha}\chi - \frac{\alpha+\beta}{1-\alpha}\delta - \frac{\alpha+\beta}{1-\alpha}\sigma H_A - \frac{\sigma}{\alpha}\frac{\alpha+\beta}{1-\alpha}(H-H_A) \tag{72}$$

$$= \frac{(1-\alpha-\beta)}{1-\alpha}\chi - \frac{\alpha+\beta}{1-\alpha}\delta + \frac{\alpha+\beta}{\alpha}\sigma H_A - \frac{\sigma}{\alpha}\frac{\alpha+\beta}{1-\alpha}H, 20 \tag{73}$$

adding and substracting $\frac{(1-\alpha-\beta)}{1-\alpha}\chi^*$ and using the first expression of χ^*, (A7), we have

$$= \frac{(1-\alpha-\beta)}{1-\alpha}(\chi-\chi^*) - \frac{\alpha+\beta}{1-\alpha}\delta + \frac{\alpha+\beta}{\alpha}\sigma H_A - \frac{\sigma}{\alpha}\frac{\alpha+\beta}{1-\alpha}H \tag{74}$$

$$+\frac{\alpha+\beta}{1-\alpha}\delta + \frac{\alpha+\beta}{1-\alpha}\frac{\sigma}{\alpha}H + \frac{\alpha+\beta}{1-\alpha}\frac{\alpha-1}{\alpha}\sigma H_A^*; \tag{75}$$

$$\frac{\dot{H}_A}{H-H_A} = \frac{(1-\alpha-\beta)}{1-\alpha}(\chi-\chi^*) + \frac{\alpha+\beta}{\alpha}\sigma\,(H_A - H_A^*)\,.25 \tag{76}$$

Secondly, consider (22):

$$\frac{\dot{\chi}}{\chi} = \frac{1-\alpha-\beta-\theta}{\theta} z + \chi - \frac{(1-\theta)\,\delta+\rho}{\theta} 22 \tag{77}$$

adding and substracting χ^* and using the 2nd expression of χ^*, (A8), we have

$$\begin{aligned} &= \frac{1-\alpha-\beta-\theta}{\theta} z + \chi - \frac{(1-\theta)\,\delta+\rho}{\theta} - \chi^* && (78)\\ &\quad + \frac{(1-\theta)\,\delta+\rho}{\theta} - \frac{1-\alpha-\beta-\theta}{\theta} z^* && (79)\\ \frac{\dot{\chi}}{\chi} &= \frac{1-\alpha-\beta-\theta}{\theta}\left(z-z^*\right) + \left(\chi-\chi^*\right).26 && (80) \end{aligned}$$

Thirdly consider (23):

$$\frac{\dot{\omega}}{\omega} = z - \chi - \delta - \sigma H_A 23 \tag{81}$$

adding and substracting z^* and σH_A^* and using the 3rd expression of χ^*, (A9), we have

$$\frac{\dot{\omega}}{\omega} = \left(z-z^*\right) - \left(\chi-\chi^*\right) - \sigma\left(H_A - H_A^*\right).27 \tag{82}$$

Finally, differentiating (21) $\dot{z}\overline{\qquad}_{z=-\alpha\frac{\dot{H}_A}{H-H_A}-(\alpha+\beta)\frac{\dot{\omega}}{\omega}}$ and introducing (25) and (27),

$$\begin{aligned} &= -\frac{\alpha\left(1-\alpha-\beta\right)}{1-\alpha}\left(\chi-\chi^*\right) - \left(\alpha+\beta\right)\sigma\left(H_A-H_A^*\right) && (83)\\ &\quad -\left(\alpha+\beta\right)\left(z-z^*\right) + \left(\alpha+\beta\right)\left(\chi-\chi^*\right) + \left(\alpha+\beta\right)\sigma\left(H_A-H_A^*\right) && (84)\\ \frac{\dot{z}}{z} &= -\left(\alpha+\beta\right)\left(z-z^*\right) + \frac{\beta}{1-\alpha}\left(\chi-\chi^*\right).28 && (85) \end{aligned}$$