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Agree or Convince

Ata Ozkaya

GALATASARAY ÜNİVERSİTESİ İKTİSADİ ARAŞTIRMALAR MERKEZİ

working papers

Agree or Convince[Υ]

Ata OZKAYA[†]

We continue the work of Aumann (*Ann. Statist.* **4** (1976), 1236-1239), Geanakoplos and Polemarchakis (*J. Econ. Theory* **28** (1982), 192-200) on common knowledge and consensus, reconsidering the arguments and the findings of both Aumann (1976) and Geanakoplos and Polemarchakis (1982) and offering different insights into the revision process. By revealing set inclusion property of the revision process, we show that the consensus conditions should be redefined. This redefinition enables us to demonstrate that until consensus is reached, in fact neither of the agents make apparent revision and each agent keeps repeating his initial posterior. Therefore the equilibrium posterior should be equal to initial posterior of the agent who does not make any apparent revision through the communication process. Our results show that regardless of the length of the communication process, it is impossible for the agents to agree on a value which is different from the initial posteriors. Finally, we shed light on some crucial points left unclear by Aumann (1976) and Geanakoplos and Polemarchakis (1982).

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Beginning from Aumann (1976), in economic theory it is well understood that opinions depend on prior beliefs held and additional information received. Aumann (1976) address the question of whether communication of beliefs between agents is sufficient to remove the differences of beliefs. Aumann demonstrates that, if two agents have identical priors and if their posteriors for a given event A are *common knowledge*, then these posteriors must be equal, even though the two agents may base their posteriors on quite different information.

Aumann did not address the issue of how common knowledge of agents' posteriors is achieved. In their elegant study, Geanakoplos and Polemarchakis (1982) address the questions left open by Aumann's argument and introduce a communication process between two agents in which the agents announce their posteriors to each other (indirect communication). Following Aumann (1976), Geanakoplos and Polemarchakis (1982) show that for an arbitrary event A, if their priors are common and if their information partitions are finite, then two agents, simply by communicating back and forth their posteriors, converge in finitely many steps to a common posterior, which is common knowledge.

In this paper, we address the same questions as Geanakoplos and Polemarchakis (1982) which have been left open by Aumann's argument. First, Geanakoplos and Polemarchakis (1982) obtain the conditions under which consensus is reached. Second, the authors show that the sum of the cardinalities of the two partitions is an upper bound on the number of steps required for the process to converge. Moreover,

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[†], Univesity of Paris-1 Panthéon-Sorbonne, Galatasaray University, ataozk@yahoo.com, aozkaya@gsu.edu.tr,

Geanakoplos and Polemarchakis (1982) introduce a special case of the revision process which can be constructed by setting the number of total steps equal to *n* (an integer) and the consensus is reached at the "final step *n*", even though no evident revision occurs until this final step.

In the present paper we provide different insights into the revision process; first, we refine the revision process and show the "set inclusion property" of the process; second, we re-determine the consensus condition and show that until consensus is reached, the agents make no apparent revisions and each agent keeps repeating his initial posterior. Therefore the equilibrium posterior should be equal to initial posterior of the agent who does not make any apparent revision through the communication process. Furthermore, our results show that regardless of the length of the communication process, it is impossible for the agents to agree on a value which is different from the initial posteriors. Third, we demonstrate that the upper bound on the number of steps required for the process to converge should be tightened. Finally, we observe that the assignment of *n* (the "final step") introduced by Geanakoplos and Polemarchakis (1982) is left indefinite and is arbitrary and their analysis does not determine the numerical value of the final step. In this sense our analysis completes theirs by adding a characterization of this numerical value. By introducing a precise definition for the "final step *n*" we show that it is not arbitrary but its determination has crucial importance on consensus analysis, in that the case they introduce is not a special case but it is indeed the nature of the revision process that no apparent revision occurs until the final step. Moreover, this final step should be characterized by the extent of the "event" on which the agents communicate.

Let Ω be a finite set of states of the world and 2^{Ω} the set of possible events. There are *H* agents, each agent *h* being endowed with a partition Π_h of Ω. When the state $w \in \Omega$ occurs, agent *h* only knows that true state of the world belongs to $\Pi_h(w)$, which is the cell of *h*'s partition that contains w. We say that a partition Π is finer than any other partition Π' if and only if for all w, $\Pi(w) \subseteq \Pi'(w)$ and there exists $w' \in \Omega$ such that $\Pi(w') \subset \Pi'(w')$. A partition Π' is coarser than any other partition Π iff Π is finer than Π'. The partition represents the ability of agent *h* to distinguish among the states of the world. The coarser his partition is, the *less precise* his information is, in the sense that he distinguishes among fewer states of the world. We say that an agent *h* endowed with a partition Π_h knows an event *E* at w if and only if $\Pi_h(w) \subseteq E$ following Aumann (1976).

To follow the setup introduced by Geanakoplos and Polemarchakis (1982), let us assume that there are only two agents, *h=1,2*. More generally, let the two partitions, Π_1 and Π_2 be finite; $\Pi_1 = \{\Pi_{11}, \Pi_{12}, ..., \Pi_{1K}\}$ and $\Pi_2 = \{\Pi_{21}, \Pi_{22}, ..., \Pi_{2L}\}$. Let *A* be an event, and let $q_h(.)$ denote the posterior probability $\Pr(A|\Pi_h)$ of *A* conditional on *h*'s information; i.e., if $w \in \Omega$, then $q_h(w) = \Pr(A|\Pi_h(w))$. In each step, the agents communicate to each other only their posteriors. Note that in order to represent the "step" we use an upper index. In step 1, agent 1 first announces $q_1^1(w)=\Pr(A|\Pi_1(w))$. Agent 2 then announces $q_2^1(w)=\Pr(A|\Pi_2(w),q_1^1(w))$. In step 2, agent 1 then announces $q_1^2(w)=\Pr(A|\Pi_1(w),q_2^1(w))$ and agent 2 announces $q_2^2(w)=\Pr(A|\Pi_2(w),q_1^1(w),q_1^2(w))$, etc. Following Geanakoplos and Polemarchakis (1982), we can introduce *Definition* 1.

Definition 1: If for some m, $q_1^{m+1}(w)=q_2^{m+1}(w)$ holds, then we say that the consensus is reached at step $m+1$.

Define the sets $\Pi_1^m(w')$ and Ω_1^m recursively as

$$\Pi_1^m(w')=\{(\Pi_1(w')\cap\Omega_2^{m-1}): \quad w'\in\Omega_1^0\} \tag{1}$$

$$q_1^m(w)=\Pr(A|\Pi_1^m(w)) \tag{2}$$

$$\Omega_1^m=\{w'\in\Omega_1^{m-1}: \quad \Pr(A|\Pi_1^m(w'))=q_1^m(w)\} \tag{3}$$

$\Pi_1^0(w')=\Pi_1(w')$ and $\Omega_1^0=\Omega$.

Define the sets $\Pi_2^m(w')$ and Ω_2^m recursively as

$$\Pi_2^m(w')=\{(\Pi_2(w')\cap\Omega_1^m): \quad w'\in\Omega_2^0\} \tag{4}$$

$$q_2^m(w)=\Pr(A|\Pi_2^m(w)) \tag{5}$$

$$\Omega_2^m=\{w'\in\Omega_2^{m-1}: \quad \Pr(A|\Pi_2^m(w'))=q_2^m(w)\} \tag{6}$$

$\Pi_2^0(w')=\Pi_2(w')$ and $\Omega_2^0=\Omega$.

It directly holds from (1) that for all $w'\in\Omega$, $\Pi_1^m(w')\subseteq\Pi_1^{m-1}(w')$ and likewise, from (4) that $\Pi_2^m(w')\subseteq\Pi_2^{m-1}(w')$, respectively. **(7)**

It directly follows from (1) and (3) that $\Omega_1^m\subseteq\Omega_2^{m-1}$ holds for all m. **(8)**

Moreover, it directly follows from (4) and (6) that $\Omega_2^m\subseteq\Omega_1^m$ holds for all m. **(9)**

The results (8) and (9) show us the "*set inclusion*" property of the revision process. It also directly follows from (8) and (9) that, for h=1,2 and $m>1$, $(A\cap\Omega_h^m)\subseteq(A\cap\Omega_h^{m-1})$.

Since Ω_1^{m+1} is set of states that are consistent with agent 1's first m+1 announcements, while Ω_2^{m+1} is the set of states that is consistent with agent 2's first m+1 announcements. Clearly, this defines two sequences of non-empty, nested sets whose limit Ω_1^{∞} and Ω_2^{∞} exist and must be non-empty.

Now, we can write $q_1^{m+1}(w)$ and $q_2^{m+1}(w)$ respectively as,

$$q_1^{m+1}(w) = \Pr\left(A \middle| \Pi_1^{m+1}(w)\right) = \Pr\left(A \middle| \Pi_1(w) \cap \Omega_2^m\right) = \Pr\left(A \middle| \Omega_1^{m+1} \cap \Omega_2^m\right) \tag{10}$$

$$q_2^{m+1}(w) = \Pr\left(A \middle| \Pi_2^{m+1}(w)\right) = \Pr\left(A \middle| \Pi_2(w) \cap \Omega_1^{m+1}\right) = \Pr\left(A \middle| \Omega_2^{m+1} \cap \Omega_1^{m+1}\right) \tag{11}$$

By using set inclusion property and Eqs. (10) and (11), we can rewrite the posteriors $q_1^{m+1}(w)$ and $q_2^{m+1}(w)$, as given in Eqs.(12) and (13) respectively.

$$q_1^{m+1}(w) = P\left(A \middle| \Omega_1^{m+1} \cap \Omega_2^m\right) = \begin{Bmatrix} P\left(A \middle| \Omega_2^m\right) & if & \Omega_2^m = \Omega_1^{m+1} \\ \\ P\left(A \middle| \Omega_1^{m+1}\right) & if & \Omega_1^{m+1} \subset \Omega_2^m \end{Bmatrix} \tag{12}$$

$$q_2^{m+1}(w) = P\left(A \middle| \Omega_1^{m+1} \cap \Omega_2^{m+1}\right) = \begin{Bmatrix} P\left(A \middle| \Omega_1^{m+1}\right) & if & \Omega_1^{m+1} = \Omega_2^{m+1} \\ \\ P\left(A \middle| \Omega_2^{m+1}\right) & if & \Omega_2^{m+1} \subset \Omega_1^{m+1} \end{Bmatrix} \tag{13}$$

According to Eq.(12), at step $m+1$ agent 1 has to announce one of two possible posteriors; one is based on his own information $\Pr\left(A \middle| \Omega_1^{m+1}\right)$, the other is the latest posterior of agent 2 which she has announced $\Pr\left(A \middle| \Omega_2^m\right) = q_2^m(w)$ at previous step m. Likewise, according to Eq.(13) agent 2 has to announce his own posterior $\Pr\left(A \middle| \Omega_2^{m+1}\right)$ or repeat the latest posterior of agent 1 which she has announced $\Pr\left(A \middle| \Omega_1^{m+1}\right) = q_1^{m+1}(w)$ at step m+1.

The preceding argument leads us to seek the conditions under which agent h has obtained new information through two subsequent steps, m and $m+1$. *Proposition* 1 deals with this issue and has crucial importance to obtain the consensus condition.

Proposition 1 :

a.) Suppose that $\Omega_1^{m+1} = \Omega_1^m$. Then $\Omega_2^m = \Omega_2^{m+1}$.

b.) Likewise, if $\Omega_2^{m+1} = \Omega_2^m$, then $\Omega_1^{m+1} = \Omega_1^{m+2}$.

Particularly, the implication of the relations given by *Proposition* 1 is as follows: if one of the agents has stopped updating his information, then no further update is possible in next steps. Eqs. (12), (13) and *Proposition* 1 enable us to identify the sufficient condition under which consensus is inevitably reached. *Proposition* 2 deals with this issue.

Proposition 2: By using Eqs.(12), (13) and *Proposition* 1, we obtain the following sufficient condition.

$$q_1^{m+1}(w) = q_2^{m+1}(w) \quad \text{holds if and only if} \quad \Omega_2^{m+1} = \Omega_2^m \quad \text{holds.} \tag{14}$$

The condition given by (14) implies that consensus is inevitably reached at step $m+1$. *Proposition* 1 and *Proposition* 2 show that at any step whenever agent 2 stops updating consensus is reached. When the condition given by (14) is satisfied, according to Eqs. (12), (13) the posterior of one of the agents should have converged to the latest posterior announced by his correspondent.

Eqs. (12), (13) and *Proposition* 2 have further implications. First, assume that during first m steps consensus is not reached $q_1^m(w) \neq q_2^m(w)$, then $\Omega_2^m \subset \Omega_1^{m-1} \subset \Omega_2^{m-1}$ for $m \geq 1$, implying that every step of the revision process the information that agent h obtains from his correspondent should change. Second, particularly if $\Pi_1^1(w) \subset \Omega_1^m$, then for all $m > 1$, $q_1^1(w) = q_1^m(w)$ holds. In words, given the condition $\Pi_1^1(w) \subset \Omega_1^m$ agent 1's first m announcements are identical and moreover, equal to his initial announcement $q_1^1(w)$. The second observation leads us to question that even though the information that an agent h obtains from his correspondent should change, the information that agent h bases on to revise his posterior changes or not? Specifically we question whether $\left(\Pi_h(w) \cap \Omega_j^m\right) \subset \left(\Pi_h(w) \cap \Omega_j^{m-1}\right)$ for h=1,2 and $j \neq h$? *Proposition* 3 clarifies this issue.

Proposition 3 : Suppose that consensus is not reached at first m steps, $m \geq 1$ and $q_1^m(w) \neq q_2^m(w)$. Then we say that in fact there have been no apparent revisions and each agent keeps repeating his initial posterior through m steps: $q_1^1(w) = q_1^m(w)$ and $q_2^1(w) = q_2^m(w)$ for all m. Furthermore, consensus should be reached in fact as soon as one of the agents makes his first apparent revision, and no further updates will be possible. Therefore, equilibrium posterior should be the initial posterior of the agent who does not make any apparent revision.

Note that our result given by (14) is the sufficient condition for the revision process to converge and completely different from the result obtained by Geanakoplos and Polemarchakis (1982) who argue that consensus is reached when both agent 1 and agent 2 gain no information.

According to *Proposition* 1 and *Proposition* 2 once an agent stops updating, no further updates are possible and agreement should be reached at that step. Since each agent can update at most K or L times, it follows that the maximum upper bound to reach consensus should be equal to $\min\{K, L\}$. Note that this result is also in contrast with the result of Geanakoplos and Polemarchakis who argue that the sum of the cardinalities of the two partitions (K+L) is an upper bound on the number of steps required for the process to converge.

It is clear from Eqs.(12) , (13) and *Proposition* 2 that at every step of the revision process until the consensus is reached, the information of each agent changes but the posterior of the agents does not change. The consensus condition determined by Geanakoplos and Polemarchakis yields possible misinterpretation in the sense that either agent 1 or agent 2 has to gain information unless the equilibrium has been reached. Note carefully that our result given in *Proposition* 2 discards such misinterpretation; i.e., if agent 1 gains information and agent 2 does not gain, then consensus is reached at that step.

Moreover, according to *Proposition* 3 each agent insists on announcing his initial posterior until consensus is reached. Therefore, it must be the case that until final step $\left(A \cap \Omega_h^m\right) \subset \left(A \cap \Omega_h^{m-1}\right)$ for $m > 1$. Consequently, we explicitly say that the indefinite term "final step *n*" can not be an arbitrary integer. We introduce *Proposition* 4 which completes the analysis of Geanakoplos and Polemarchakis (1982) and shows that it is not a special case but it is indeed the nature of the revision process to take $n \geq 1$ steps and no apparent revision occurs until the final step. Moreover, this final step should be characterized by the extent of the "event" on which the agents communicate.

Proposition 4: If we want to construct a revision process which takes *n* steps to converge then no apparent (evident) revision occurs until the final step *n*, and that the number of total steps – final step (n) can not be an arbitrary integer but should be set equal to the cardinality of event A, # A.

References

[1] Aumann, R.J., (1976), "Agreeing to disagree", *Annals of Statistics* 4, 1236–1239.

[2] Geanakoplos, J. D. and Polemarchakis, H. M., (1982), "We can't disagree forever" *Journal of Economic Theory* 28, 192–200.

Appendix

Proof of Proposition 1:

a.) If $\Omega_1^m = \Omega_1^{m+1}$, then $q_2^{m+1}(w) = \Pr\left(A \middle| \Pi_2(w) \cap \Omega_1^{m+1}\right) = \Pr\left(A \middle| \Pi_2(w) \cap \Omega_1^m\right) = q_2^m(w)$, and for all $w' \in \Omega_2^m$, $\Pr\left(A \middle| \Pi_2(w') \cap \Omega_1^{m+1}\right) = \Pr\left(A \middle| \Pi_2(w') \cap \Omega_1^m\right) = q_2^m(w) = q_2^{m+1}(w)$ implying $w' \in \Omega_2^{m+1}$, and therefore $\Omega_2^m = \Omega_2^{m+1}$.

b.) Likewise, if $\Omega_2^m = \Omega_2^{m+1}$, then $q_1^{m+1}(w) = \Pr\left(A \middle| \Pi_1(w) \cap \Omega_2^m\right) = \Pr\left(A \middle| \Pi_1(w) \cap \Omega_2^{m+1}\right) = q_1^{m+2}$, and for all $w' \in \Omega_1^{m+1}$, $\Pr\left(A \middle| \Pi_1(w') \cap \Omega_2^m\right) = \Pr\left(A \middle| \Pi_1(w') \cap \Omega_2^{m+1}\right) = q_1^{m+1}(w) = q_1^{m+2}(w)$, implying $\Omega_1^{m+1} = \Omega_1^{m+2}$.

Proof of Proposition 3: Suppose that through $m \geq 1$ steps consensus is not reached, $q_1^m(w) \neq q_2^m(w)$. For step *m*+1, according to Eq.(10) the belief of agent 1 is shown as $q_1^{m+1}(w) = \Pr\left(A \middle| \Pi_1(w) \cap \Omega_2^m\right)$. Therefore, for all $w' \in \left(\Omega_1^{m+1} \cap \Omega_2^m\right)$ $\Pr\left(A \middle| \Pi_1(w') \cap \Omega_2^m\right) = q_1^{m+1}(w)$ have two possible values;

a.) if $\Omega_1^{m+1} = \Omega_2^m$ then $\left(\Pi_1(w') \cap \Omega_2^m\right) \subset \left(\Omega_1^{m+1} \cap \Omega_2^m\right)$ implying $\left(\Pi_1(w') \cap \Omega_2^m\right) \subset \left(\Omega_1^{m+1} = \Omega_2^m\right)$,
and according to Eq.(12) $q_1^{m+1}(w) = \Pr\left(A \middle| \Pi_1(w') \cap \Omega_2^m\right) = \Pr\left(A \middle| \bigcup \Pi_1(w') \cap \Omega_2^m\right) = \Pr\left(A \middle| \Omega_2^m\right) = q_2^m(w)$, implying consensus.

b.) if $\Omega_1^{m+1} \subset \Omega_2^m$ then $\left(\Pi_1(w') \cap \Omega_2^m\right) \subset \left(\Omega_1^{m+1} \subset \Omega_2^m\right)$, and according to Eq.(12)
$q_1^{m+1}(w) = \Pr\left(A \middle| \Pi_1(w') \cap \Omega_2^m\right) = \Pr\left(A \middle| \bigcup \Pi_1(w') \cap \Omega_2^m\right) = \Pr\left(A \middle| \Omega_1^{m+1}\right) \neq q_2^m(w)$.

Since, the true state should be $w \in \left(\Omega_1^{m+1} \cap \Omega_2^m\right)$, according to Eq.(10) for $q_1^{m+1}(w) = \Pr\left(A \middle| \Pi_1(w) \cap \Omega_2^m\right)$ three cases are possible;

a.) if $\left(\Pi_1(w) \subset \Omega_2^m\right)$ then $q_1^{m+1}(w) = \Pr\left(A \middle| \Pi_1(w) \cap \Omega_2^m\right) = \Pr\left(A \middle| \Pi_1(w)\right)$, implying $q_1^{m+1}(w) = q_1^1(w) \neq q_2^m(w)$.

b.) if $\left(\Pi_1(w) \cap \Omega_2^m\right) \subset \Pi_1(w)$ and $\left(\Pi_1(w) \cap \Omega_2^m\right) \subset \Omega_2^m$ then for all $w' \in \left(\Omega_1^{m+1} \cap \Omega_2^m\right)$
$q_1^{m+1}(w) = \Pr\left(A \middle| \bigcup \Pi_1(w') \cap \Omega_2^m\right) = \Pr\left(A \middle| \Omega_2^m\right)$ since because $\left(\Pi_1(w') \cap \Omega_2^m\right) \subset \Omega_2^m$. Thus, implying
$q_1^{m+1}(w) = q_2^m(w)$.

c.) if $\left(\Pi_1(w) \supset \Omega_2^m\right)$, then $q_1^{m+1}(w) = \Pr\left(A \middle| \Pi_1(w) \cap \Omega_2^m\right) = \Pr\left(A \middle| \Omega_2^m\right)$, implying $q_1^{m+1}(w) = q_2^m(w)$.

The same analysis is also true for agent 2.

Proof of Proposition 4: Assume that consensus is reached at step m+1. According to *Proposition* 2, for all $m > 1$ and h=1,2 $q_h^1(w) = q_h^m(w)$. Thus $\frac{\Pr(A \cap \Omega_h^1)}{\Pr(\Omega_h^1)} = ... = \frac{\Pr(A \cap \Omega_h^m)}{\Pr(\Omega_h^m)}$ and $\Omega_h^m \subset \Omega_h^{m-1}$, implying $(A \cap \Omega_h^1) \supset (A \cap \Omega_h^2) \supset ... \supset (A \cap \Omega_h^m)$. Therefore if we assume that at each step at least one element of A is eliminated, then the maximum number of steps required to reach consensus should be equal to the cardinality of A, # A = m+1.